Calculus & the Mind of God

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$$e^{i*\pi} + 1 = 0$$

$\therefore$ *God* $\exists$

– Leonhard Eüler

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An Introduction: Mathematics & Theology

In undertaking a study of the Calculus, one finds oneself in the realm of pure mathematics. It is not the specialist mathematician, however, but the mathematician-theologian[1] who asks if mathematics bears upon theology. Every so-called "mathematical proof" of God's existence is inscrutable or ludicrous, but this does not mean that there is no relation between mathematics and theology. The Psalmist says, "The heavens show forth the glory of God, and the firmament declareth the work of his hands."[2] Mathematics and Calculus, then, as parts of creation, must glorify God.

The method of the Calculus has more than a cursory connection to theology. In his *Summa Contra Gentiles,* Saint Thomas describes the "way of remotion." he calls it an "approach" towards knowledge of God.[3] The method of this approach is paralleled by the method of knowledge of the the limit in Calculus.

In order to show this, we must first consider, in depth, the concept of the limit, although we must not develop an understanding that is strictly metaphysical[4] or specifically

1 Or the liberal arts student...
2 Psalm 19:1
3 *Summa Contra Gentiles,* Chapter 14
4 See Appendix A.

mathematical.[5] Instead, we must pursue an understanding sufficient for the interest of the mathematician-theologian.

First, with an eye to the theological, we look at limits in physical things, and then in mathematical things. We inquire into motion in mathematics and see how the limit uses that motion. We develop a general expression for the notion of the limit. Next, we discern how infinity relates to the limit in its various forms, and then how the Thomistic concept of "relative infinity"[6] bears upon the limit. We develop the relative infinite as an explanation for the workings of the limit concept itself by using it to explain how a rational nature is able to surpass mathematical infinity.[7] With this knowledge of how the infinite operates within the limit, we carefully examine the knowledge that the limit produces within Calculus. We investigate whether infinity itself may be a limit within mathematics. This sets the foundation for the limit concept to be elevated from mathematics to theology.

In the third part of the thesis, we actually think about the way of remotion within the structure of the limit we developed from mathematics. We distinguish two senses of infinite, one that applies to quantity and one that applies to God. Analogously to the limit of an infinite series, God stands as a limit to the infinite process that makes up the way of remotion. We conclude that, within the way of remotion, He is the logical end to each motion that is towards Him,[8] and that with each motion, and similarly with the whole way of remotion, He is reached logically as the limit of the approach but He is not reached in Himself as the limit of understanding.

We then end the discussion with a brief introduction to the logical names for God that come out of the way of remotion, and with a brief comment that the real end of our approach towards knowledge of God is outside our reach in this life, but is promised to us in the Beatific Vision: it is nothing other than a sharing in God's knowledge of Himself.

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5 See Appendix B.
6 *Summa Theologica*, Q50, A2, Reply Obj. 4
7 A mathematical operation, though infinite in itself, stands as finite to a logical operation.
8 *Summa Contra Gentiles,* Chapter 14

Part I: What is a Limit?

A complete consideration of the limit is necessary. Mathematics is the science of quantity. Because of what quantity is, man understands the truth of the science with ease and exceptional clarity. The truth in mathematics unfolds smoothly and logically from the principles of the science. It is beautiful to behold. Man's intellect clearly grasps the causes, the effects, the middle terms – the entirety of the propositions of the science. Philosophers and metaphysicians repeatedly refer to mathematics as the epitome of the sciences. Though particular human intellects may vary in aptitude for mathematics, human nature itself seems suited to pursuit of mathematical knowledge. Throughout history, each new mathematical discovery has built upon the previous foundation of knowledge, allowing more and more complicated proofs and theorems to develop. No other science is as straightforward as that of quantity. No other science exists where man's unaided intellect can so clearly see one truth unfolding from another.

Though the objects of mathematics are operated on by the mathematician in the imagination, quantity truly exists in the material.[9] In fact, the material realm is where quantity exists *primarily*. Providentially, man derives his natural knowledge from this same material existence.[10] Man observes quantity housed in the familiar natural substances which surround him. From these, he abstracts at least the principles of mathematics, if not some higher truths – the Pythagorean Theorem, for instance. Once the principles of mathematics or quantity are in the intellect and are possible subjects for the imagination, man may then expound truth from truth, not creating new truth, but rather discovering truths previously unknown to him or possibly unknown to mankind in general.

9 *Summa Theologica,* Question 7, Article 3, Reply to Obj. 3
10 Quantity as an accident inheres in substance.

The Limit in Three Senses

Man naturally receives knowledge into his intellect through the mediation of his bodily senses. Since man's knowledge of quantity is primarily natural knowledge, this too enters into the intellect by way of the senses. Quantity really exists in sensible things, and man knows the pure form of quantity – the one that he deals with in mathematics – by the power his mind has to abstract quantity from the accidents of sensible things. This quantity in man's intellect, separated from the substantial beings in which it first coexists, forms the basis of mathematics. Because man knows quantity as from the material, it is logical, when explaining a mathematical concept, to look at material experience in order to clarify the concept and expand its definition. Thus, in pursuit of the limit, we first look at the material world from which the concept arises.

It is manifest that all created things are absolutely finite.[11] When we observe a material thing, we generally observe it as finite. At a certain place in space,[12] the object stops and everything else starts. There is an edge, or boundary to each object. With hard material bodies, they carry these boundaries with them, as fixed. Other bodies, such as liquids or fires also have boundaries, although fluctuating ones. The boundaries of things influence our perception of those things, and inform us as to their finitude by means of our senses. Boundaries are, in a way, what makes bodies finite. They *define* what they belong to, at least as far as they affect our perceptions of them. That which defines, or renders finite, limits. Thus, the first and most immediate sense of "limit" is synonymous with "boundary."

The next sense of limit is that which belongs to motion. Motion terminates in a boundary or limit analogously to the way that both material things and abstracted first

11 That is, absolutely *speaking*.
12 These physical terms are here used in a everyday sense. A complete discussion of place and space is not necessary here, nor is it immediately relevant to the topic at hand.

dimension quantity do, since the boundaries or limits of extension underlie motion. However, this definition of limit bifurcates as well. Two senses of the limit of motion proceed from looking at motion as actual or possible.[13]

Actual motion has a limit contained virtually within it as the mobile moves; this limit is apparently actually existent at the end of the motion, coming into its relative being at the instant of that motion's cessation. Possible motion has a limit as defined by the underlying quality or nature of that motion and it is understood logically by an intellect which excels the motion in essence. The simplest example of this underlying quality or nature is that of the magnitude which underlies linear motion. In a mathematical operation, say motion along a line, that underlying quality is the magnitude which it traverses, and the limit or boundary of that magnitude (i.e. limit in the first sense) dictates the limit of motion, or the logical terminus of any possible motion.

If a man is climbing a mountain, the limit of his expedition is the top of the mountain: he cannot continue his same motion beyond that point. This reality is not changed by the actuality of the ascent. It is a reality possessed by the magnitude that underlies any possible motion up the mountain. Even if the man never makes it past the trailhead, the mountain's summit still stands as the limit of any climbing of that mountain, past, present, or future.

Man knows that the summit must be the terminus of any ascent, and he knows this by a logical operation.[14] He knows it from the reality of the mountain, but through his reason. Thus, the peak of the mountain can be a limit under all three meanings of limit. First, as the limit of the physical extension of the mountain – that is, the boundary of the mountain or its height; second, as the limit of a particular actual motion up the mountain;

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13 I say "possible" here in order to highlight the *conditional* nature with which we are dealing, and to ensure that we are not simply taking the *potential* that is the correlative of the our previous "actual."

14 An argument for the intelligibility of nature to any sensate animal being seems to require the outline of a complete epistemology; thus, it is outside the scope of this paper.

and third, as the limit of any possible ascent of the mountain. The difference between the latter two limits is not as subtle as it may seem.

Are these limits reached? The first sort of limit is reached simply because the substance to which it belongs exists. When, however, the limit is considered as the *limit of motion,* the question of whether it is attained or not is meaningful. There may be nothing that distinguishes the successful and unsuccessful attempts to climb the mountain below the treeline – they are equally *motion towards* the summit. The summit is virtually contained in each of the actual motions, but whether the motions succeed in reaching the summit is dependent upon whether these motions each exist as sufficient to do so, and whether external circumstances block this sufficiency.

Thus we have a distillation of the notion of this third meaning of limit. The limit, under this consideration is **the logical terminus of an operation**. This does not specify the mathematical terminus of a particular operation, though the two sorts of ends may accidentally be the same thing. It does not even mean the mathematical termini of a number of or all of the operations, for indeed as we shall see many operations of which we comprehend limits do not possess mathematical termini of themselves.

Such a limit cannot be completely understood through the mode of the operation. For the operation is necessarily of a different kind than the operation which defines the limit. When trying to understand the limit through the mode of the operation, it can only be understood as that beyond which the operation may never pass. Though the mind grasps the limit as the necessary end (in a way) of the operation or motion, in so doing it necessarily surpasses the original operation. This true understanding comes not from delving into the mode of the operation but rather from understanding of a higher order. The necessary end of the motion comes from *what the motion is;* it is understood only by the intellect's grasp of the motion's quiddity. This is only possible because of the intellect's higher order of being: it is relatively infinite to even the infinite in quantity or mathematics.

It is clear that the existence of the limit does not depend on any actual performance or end of the motion, but rather upon what underlies it. It is merely the logically perceived and understood end of the motion.

The Limit in the Physical

Just as mathematical quantity differs from the quantity present in material things through the process of abstraction, the mathematical limit differs from limits found in physical things. We frequently encounter physical limits in simple motions.[15] In walking across a room, the room's opposite wall is a limit to that motion. The motion cannot proceed past the wall, at least not without changing into a different form of locomotion – destruction of the wall, or of the perambulator, for example. But the status of the wall as limit does not depend on the actuality of the motion. Even if the motion ends before it reaches the wall, the wall is still a limit to that *kind* of motion. Any linear motion within the room must terminate at that wall. Thus, if the motion in question is perambulation within the room, the walls are present in that motion as virtual limits.

Consider another physical example. Starting with a whole apple, take away half. From the half that remains, take away another half. This successive subtraction of halves *mathematically* never ends. Yet in the physical reality even the child immediately knows that *no more than the whole* can ever be taken away. Thus the limit of the quantity subtracted is the whole, and the limit of the remainder is nothing. This is fundamentally certain. Even if the operation terminates before this point – and if the subtraction occurs a finite number of times it must – such limits remain, nonetheless, fixed. They exist independently of the operation, drawing their existence only from that which underlies and informs the possible motion. The mathematical limit is never reached, yet every child knows what the *logical* limit must be.

15 I will, at present, limit the considerations to linear motion.

The Limit in Mathematics

These three meanings of limit are readily transcribed from the physical to the mathematical realm. The first sense of limit, the simple sense of boundary, is that which accompanies finite first-dimension quantity. We are speaking of a geometrical boundary of the simplest kind and order: the boundary between two lines that meet at a point.

A point is that which has no part. It is the boundary of a line. Two line segments AB and BC both share the point B as their common boundary. Who is to say that the point belongs more to one line than the other? The point B is the last point of AB and the first point of BC. At the point, it clearly cannot be both of AB and BC simply, for if it were, it would stand in exactly the same relation to both. Rather, by naming B as the last point of AB and the first of BC, it is named the only modes in which it can be known. These two modes are the only ones in which it can be something of both AB and BC, where both things coexist and differ. If B were a line, a finite line, it could not be part of two distinct line segments; rather, it *only* could exist with relation to two line segments as part to whole. Thus, the *mode of existence* here is the key thing.

Existence admixed with being that can be multiplied *at will* is necessarily relation. Relation is existence with reference to another; thus, relation is produced, or generated, or at least comes into existence, without modification of the original being, but merely with the noticed existence of another.

In the previous example, the point B may be the end point of an infinite number of lines, a pencil of lines, each with B being the terminus of some segment. Thus, the point may have infinite modes of existence, all relations existing with reference to some other. It is the last (or first) point in the lines AB, CB, DB, EB, and so on, ad infinitum. These boundaries are all *limits* of their respective line segments, but in itself, the limit is simply one and the same point, B.

The Limit in Mathematics: Motion

Boundaries are intelligible, but their comprehension demands some sort of motion. This is true in mathematics, but even the human modes of perception make use of motion. Sight, for example, uses motion in one way when the eye moves around the room in order to cover objects and take them in, and in another way which underlies all sight: the motion of light itself. The human eye does not see distinctly what it looks straight at; instead, the brain uses motion to fill in the gaps and give the picture perspective. Without motion, the edges between things are not distinct or necessarily apparent. Even speaking of the boundary as "where one thing stops and another starts" presumes action on the part of the existence of the objects. All of our senses are similarly reliant on movement, either of the sensible or sensate matter, or of the one who senses. Thus, all knowledge of boundaries gained through the human senses relies on motion.

Motion in mathematics is different from physical motion because all motion in mathematics is enacted solely by the will of the mathematician. It is clear, then, that the first meaning of limit – the one that is synonymous with boundary – has motion tied up with it and its intelligibility. The second and third types of limits have motion explicitly stated in their definitions; the first as the terminus of the actual motion, the second as the logical terminus of the possible motion.

When mathematics moves from the first notion of the limit – the boundary of finite first dimension quantity – to the second or third notion of limit, the involvement of motion becomes explicit. The first kind of limit is "where the object ceases its extension," but the second is the cessation of the object's movement. The first is limit with reference to magnitude or extension, the second is limit with reference to motion and all the things that are requisite for a discussion of motion.

Mathematicians typically speak of "operations" instead of motions, but Saint Thomas speaks of the way in which every operation is a motion.[16] Thus, the limit in this sense bounds motion in mathematics; this is analogous to the second sense of limit that we developed above for physical mobiles. The next step in this discussion is the crucial shift from the second notion of the limit to the third notion; that is, from the use of the limit as an end virtually contained in actual motion, to the full concept of the limit as used by the Calculus, whether the limit is the logical terminus of a possible operation.

This brings up the question of motion towards. Motion may be towards, or "in the direction of" a plurality of destinations simultaneously. Indeed, it may be toward an infinity of points without having a single one of that set being the actual destination. Further, motion may be towards something without being sufficient to reach that something on its own, as, for example, a weak man may try, unsuccessfully, to scale a mountain. Motion may be towards something and sufficient to get there under normal circumstances, but not all-powerful in getting there, for example a fit mountain climber who gets stranded by a storm.[17]

An example of motion in mathematics may be taken from the consideration of an infinite series. In the infinite series { 1 + ½ + ¼ + 1/8 + 1/16 + ... }, the sum is moving towards, or in the direction of, 3 just as much as it is moving in the direction of 2.[18] Each time it moves closer to 2, it also moves closer to all numbers greater than 2. For that matter, it is just as much moving towards 10,000 or any other finite number that is larger than 2. However, in the nature of the operation, something distinguishes the number 2 as the logical terminus of the operation. The sum can get as close as you please to 2, without ever surpassing it. It is the first thing that it can never reach through its own operation. It is clear that this has to be a limit in the third sense, because there is no way for any actual motion to be completed, even in the intellect of a rational nature.

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16 *Summa Theologica,* Question 9, Article 1, Reply to Obj. 1 & Question 73, Article 2, Body

17 Only human pride presumes that both the end and sufficiency towards that end exist completely in our own actions.

18 It "moves" as a mathematical motion solely because of the will of the mathematician.

Part II: Infinity within the Concept of the Limit

Once we have this complete outline of the meanings of limit, both in physical reality and in mathematics, it remains to be seen how the concept of the limit involves infinity. The concept of the limit is hard for the human mind to conceive, for it is based not only upon the infinite, but also upon motion through the infinite. The concept of the limit fundamentally contains *a logical traversal of the infinite*.

At the outset, this appears to be an impossibility, a contradiction, for the infinite, as infinite, is unknown, as even Aristotle knew.[19] Saint Thomas expounds upon this. He says, "Again, the infinite cannot be known in so far as it cannot be numbered."[20] However, Saint Thomas goes on to lay the foundation that is necessary to resolve this seeming contradiction. He explains the *relative infinite.*

> Every creature is absolutely finite, inasmuch as its being is not absolutely subsisting, but is limited to some nature to which it belongs. But there is nothing preventing a creature from being considered relatively infinite. Material creatures are infinite on the part of matter, but finite in their form, which is limited by the matter which receives it. Immaterial created substances, however, are finite in their being, but they are infinite in the sense that their forms are not received in anything else. This would be as if we were to say, for example, that whiteness existing separately is infinite as regards the nature of whiteness, since it is not contracted to any one subject; while its *being* is finite as determined to some one special nature.
> That is why it is said that *intelligence is finite from above,* as receiving its being from above itself, and *infinite from below,* as not received in any matter.[21]

This means that, for example, a finite immaterial thing is relatively infinite when compared to a material thing, even if this thing were materially infinite.

The Infinite in Physical Reality

Though we do not think of it in everyday life, the infinite underlies all limits. Sticking to the realm of material limits, we must discuss how the infinite is present in the three

19 Aristotle, *Physics,* Book I, 187b7
20 *Summa Contra Gentiles*, Chapter 69, Paragraph 11, p.229
21 *Summa Theologica*, Q50, A2, Reply Obj. 4

material forms of limit. We have already discussed above how motion is necessary to the intelligibility of boundaries, and the subsequent meanings of limit have motion explicit in their definition. Thus, the question becomes an inquiry into how the infinite underlies physical motion.

Every finite motion that exhausts finite magnitude in finite time traverses the infinite divisibility of that underlying continuous magnitude. Obviously, traversal of the infinitely divisible is not a contradiction in terms, for it occurs in literally every instant of this material world's existence. How can this be? It can be because motion is on a different order of being than extension. Motion through the continuous magnitude is possible because the being which directs it is on a higher order than the magnitude itself. The magnitude, though infinity divisible insofar as it is magnitude, is finite relative to linear motion.

Thus, a length, AB, has no simple[22] ratio to the square on that length, because AB^2 is in another dimension. It is second dimension quantity, or area. As such, it is relatively infinite to the finite length AB. Even if AB were unlimited or infinite in its one dimension, it would still be relatively finite to AB^2, and AB^2 would still be relatively infinite to it, because, as first dimension quantity, AB simply lacks that second dimension. Thus, even the mere existence of the infinitely divisible shows that its extension is relatively infinite to that which it would be divided into, for that into which it is infinitely divisible exists only in potency.

Though I have claimed that the infinite underlies the concept of the limit, the presence of the infinite in examples from everyday experience generally escapes notice because we are used to dealing with quantity in the full order of three dimensions. Thus, infinitude in lower dimension quantity regularly escapes our notice because it stands as relatively finite to us.

Going back to the example of straight-line motion confined in a room, within the room the pedestrian passes through a *continuous magnitude,* the finite first dimension

22 It can, of course, be compared by application of areas. However, this is not pertinent to the argument that these geometric objects differ in dimension.

quantity that underlies the extension of the room. In the process, he passes through an infinity of points, since between any two points that lie on a continuous magnitude there are always an infinite number of points. Clearly this can only be possible in virtue of a motion that has a higher order of being than the extension which it exhausts. Every traversal of something which is infinite in itself, but relatively finite follows this same path.

The Infinite in Mathematical Limits

A limit in mathematics is reached only by motion, but not by a *mathematical* motion. It is reached by an act of understanding that comprehends the end of the infinite mathematical motion. As such, it can only come from an operation that is higher than the mathematical one. To think of this understanding as a motion is not altogether inaccurate, but it tends to obscure the distinction which must be made concerning the mathematical motion. The limit provides a means to traverse the infinite in understanding, but not in itself. This means is not a mathematical motion. It only can be an operation or motion of a higher order: a logical one. The motion that produces knowledge of a mathematical limit has been described as a "leap of the mind," a leap that passes through every possible mathematical configuration of the operation in question instantaneously. This "leap through" the mathematical, then, is done by logical reasoning, which is the proper operation of a rational nature.

It is important to note that the infinite is not traversed in itself by an operation that attempts to proceed through it. It is manifest that no operation is able to proceed through an infinite number of terms. The operation under discussion is a mathematical operation and mathematically never ceases through its mathematical definition.

Despite all this, that which was infinite *is* traversed. As traversed, it is clearly not infinite, and yet it retains everything it had prior to traversal. Because it was traversed in virtue of a relation – something that does not add or subtract being by modification of what already exists – we know that, in itself, it remains the same. That which was infinite has been exhausted, for the limit is found and understood. This traversal is accomplished by the intellect, by the reason and the will of the mathematician. It is a reduction from a consideration of something which is infinite in itself to a consideration where it is finite, for something other than it stands as infinite. This operation is able to be accomplished by an intellect, a rational nature. Such a nature essentially exceeds lesser being, for example, the being of quantity and motion through it, in such a way as to be able comprehend it.[23] Thus, though the rational nature is not able to proceed through the infinite directly, that is, mathematically in this case, yet it is able to proceed through the mathematical operation in virtue of a higher rational act – a logical operation. For the rational is relatively infinite to quantity, even quantity in motion.[24]

The order of relative infinites is easily understood within geometry. The point, being that which has no part, in itself is finite, but there are an infinite number of points in a finite line. Likewise, there are an infinite number of lines in a finite area, and there are an infinite number of slices of area in a cube. This is because the line exists in a higher *dimension* than the point, and the area than the line, and the cube than the area. Each class of objects stands to the objects in a class one higher as relatively finite, and each higher class of objects stands as relatively infinite to those classes below.[25]

Knowledge Produced by the Limit in Calculus

Knowledge produced by the limit in Calculus is not necessarily ordered to the infinite, or to knowledge of infinite mathematical objects. Calculus is full of operations and problem

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23 Whether an intellectual or rational nature of this sort must participate in immateriality for this to be possible seems to be different subject, not able to be treated here.

24 In another way, this is clear because the objects of mathematics are imagined by the rational nature in the first place; that is, they are already dependent on the rational nature for their existence. Understanding this causality is enough to shows the relative finitude of the mathematical objects.

25 It is important to note, however, that all mathematical objects are absolutely finite, for their forms are terminated.

solving in which the limit is employed to traverse a particular infinite yet yield a finite answer. For example, by the use of the limit in Integral Calculus, the mathematician may calculate the finite area bounded by two curves.

First, the area is divided into a finite number of segments. Then, an equation for one of these segments is reasoned to, and then that equation is generalized so that it represents every segment. Next, the expression is evaluated for an increasing number of segments. Finally, the limit is taken as the number of segments approaches infinity, and an answer – a finite answer, perhaps – is found for the bounded area.

The evaluation of finite areas with infinite boundaries is also possible through the use of the limit. For example, if the value of **e**, the base of the natural logarithm, raised to the horizontal variable **x**, is graphed on a Cartesian coordinate system grid, the curve or function exists for all values of **x**, from **-∞** to **+∞**.

Using the method of Integral Calculus, the area from **-∞** to **0**, the origin, can be evaluated, and the area turns out to have a value of **1**, negative because it extends to the left of the origin. Thus, an infinite method is used to measure a finite area that has an infinite bound.

The limit in Calculus also allows man to evaluate ratios that previously were unintelligible. For example, the ratio **1/∞** is intelligible only when the conception is that some ratio, **1/x**, where **x** is increased without limit – towards infinity. Thus, the mind grasp that as **x** grows, the ratio **1/x** shrinks, and this simultaneous resizing produces the logical termini of infinity for **x**, and **0** for the entire ratio. Thus, **1/∞**, or, by extension, the ratio of any finite quantity to infinity, evaluates as zero.

Similarly, the expression **∞/1** evaluates to infinity, showing that, even mathematically, finite being can have no ratio to infinite being. This is done by the same process. As **x** in **x/1** grows towards infinity, the ratio **x/1** increases. Thus, the logical end of **x** is infinity, and similarly for the ratio **x/1**.

A third example is the ratio of **1/0**, being to non-being. Normally in conventional mathematics, division by zero is meaningless or absurd. However, through the use of the limit, it can be evaluated. If the beginning ratio is **1/x**, then as **x** goes to **0**, the value of the ratio **1/x** increases. Thus, one can see that the logical terminus of **x** is **0 –** for it can get no smaller – and the logical terminus of the ratio is **∞**.

Is Infinity a Limit?

Consider, then, whether infinity itself can be a limit. For example, in the infinite series { 1 + 2 + 4 + 8 + 16 + ... } there is no finite number which logically concludes this operation. It appears that the infinite cannot be a limit of something because "limit" implies finitude. However, this meaning of limit transcends the etymological meaning of the word. The logical terminus of an operation as such may indeed be infinity. Remember that in order to know or attain this terminus, infinity must have already been traversed. This is why, in the solution of these problems, an appeal to a higher order of being is necessary. This allows the infinite in quantity to be considered as relatively finite, and thus the problem of traversal of the infinite in itself is circumvented.[26] Though it may be relatively infinite to other members of quantity, it stands as relatively finite to immaterial being. Thus, the rational nature, perceiving infinite quantity as relatively finite, may traverse it with ease. Hence, an "infinite limit" is not a contradiction. The infinite can indeed be a limit in this way. Thus, even in mathematics, infinity may be seen by a rational nature as the logical terminus of the operation.

Man's Knowledge of the Infinite

The infinite in itself is unknowable, but this does not mean that man cannot know

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26 *Summa Contra Gentiles,* Chapter 69. This is how God knows the infinite (and all intellectual natures, but of course, not univocally). In this way, the infinite is not known *as infinite*, but as finite, because it stands as *relatively finite*...

things about the infinite, or even about infinite things. The discussion of the infinite often seems problematic, and in such discussions meticulous attention must be paid to the specific meaning of the terms. Common errors arise because the infinite is treated as something knowable in itself. Generally, either some particular finite predicate is attached to it, or it is treated as a number in some different way, instead of it being correctly understood as opposed to the numerable. These are not difficult mistakes to make; nonetheless, infinity *can* be dealt with in the mathematical realm. It simply requires far more analytical thought than rote manipulation does.

Man knows something even of infinite quantity from his study of limited quantity in mathematics and the Calculus greatly expands that knowledge. Further, it is primarily by use of the limit in the Calculus that man wrestles with and comes to an understanding of the relative infinite.

Part III: The Limit as a Theological Construct

In order to translate the concept of the limit to Theology we must first understand what the limit is in physical reality and then in mathematics. Then we must understand what knowledge the limit produces within Calculus and how that knowledge is obtained. We must then understand how the infinite is used in the concept of the limit. The first two sections of this thesis have been devoted to these ends. In this final section, we must strive, first, to understand the mode in which we know God in this life, and, second, to see if the concept of the limit may be brought to bear on this mode and on our actual knowledge of God.

Our mantra has been that the infinite in itself is unknowable. Since God is absolutely infinite, superficially it would seem that there can be no knowledge of God.[27] However, Saint Thomas affirms that we do indeed know something of God, but that our knowledge is not direct. Instead, we know Him in a limited way, through His effects. Man first understands creatures and then approaches understanding God. In the words of Saint Thomas,

> [B]y its immensity, the divine substance surpasses every form that our intellect reaches. Thus we are unable to apprehend it by knowing *what it is.* Yet we are able to have some knowledge of it by knowing *what it is not.* Furthermore, we approach nearer to a knowledge of God according as through our intellect we are able to remove more and more things from Him.[28]

Transferring the concept of the limit from Calculus to Theology is a delicate matter, but Saint Thomas here gives us hope. He says that we may have "some knowledge" of the divine substance, and he even uses the same language that the mathematicians use to describe limits when he discusses our "approach nearer" to knowledge of God.

The first important point to establish when comparing the limit to our knowledge of God is to realize that there are two types of infinite involved. The first is the infinite taken in the privative sense, which is unique to quantity, and the second is the infinite taken in the

27 *Disputed Questions on Truth,* Volume 1, Question 2, Answer 5
28 *Summa Contra Gentiles,* Chapter 14, Paragraph 2

negative sense. It is this latter sense in which it is said of God.[29] Saint Thomas says that to be infinite in quantity signifies an imperfection, since "this quantity is of a nature to have a limit." In fact, the infinite does not actually exist within the category of quantity.

For God, however, "the infinite is understood only in a negative way, because there is no terminus or limit to His perfection."[30] Thus, attributes of the infinite in quantity do not necessarily translate to the infinite of the divine realm. Extreme care in thought and analysis is necessary, then, to modulate from the quantitative realm to the theological one.

The reality of the divide between these two realms and two types of infinite becomes clear when we focus on God and the categories of being of created things. God, Who is Being Itself, is not comprehended under any of the categories and is not confined within any genus, but mathematical infinity is inescapably linked to the genus of quantity. Strictly speaking, it is not existent in it, for it only exists in potency; that is, it exists insofar as quantity inheres in matter which admits of infinite division. An actual infinite magnitude cannot exist, as Saint Thomas proves:

> Although the infinite is not against the nature of magnitude in general, it is against the nature of any species of it. ... Now what is not possible in any species cannot exist in the genus, and hence there cannot be any infinite magnitude, since no species of magnitude is infinite.[31]

Mathematical infinity is therefore only known privatively, because it is "in the nature of quantity to be limited."[32]

To predicate the infinite of quantity of God, then, would be to relate Him to a category, not as its principle, but rather as an privative whose existence is limited to potency. This existence is opposed to the divine nature fundamentally, for God possesses the perfections of every category without being comprehended under any of the categories themselves. He is, instead, the principle of every category and the measure of all.[33] Indeed,

29 *Disputed Questions,* Volume 1, Question 2
30 *Summa Contra Gentiles,* Chapter 43, Paragraph 3
31 *Summa Theologica*, First Part, Question 7, Article 3, Reply to Objection 2
32 *Summa Contra Gentiles,* Chapter 43
33 *Ibid.,* Chapters 25 & 28

He exists as perfectly completed. Thus, the infinity of God must transcend the "mathematical" designation and not be constrained by any relation to quantity.[34]

Hence, He *is not actually infinite* within quantity, not in number, multitude, or magnitude. Therefore, God cannot strictly speaking be the limit of any mathematical equation, operation, or function. However, mathematical functions do parallel His existence and our understanding of His existence by analogous operation. This is how the concept of the limit itself transcends its mathematical, and formerly physical, origins, and translates to theology.

Application of the Limit to Theology; That God Stands as a Limit that is Reached

By using what we have said above – that infinity may stand as a limit in the third sense to an operation – we can see that it is possible that God might be the logical terminus of that operation of our knowledge which approaches Him without equivocating on infinity. God may be this logical terminus in two ways.

If our method be the "way of remotion," through which we "approach" God, the operation is then that of understanding finite creation and its perfections and moving from these to God. A motion of this kind is analogous to the limit in Calculus. It involves a never-ending motion, which is analogous to a mathematically-defined perpetual motion. Further, the terminus of this theological operation cannot be reached by proceeding through it. It can only be reached by a higher operation, much like the mathematical operation in the limit in Calculus is not reached mathematically, but logically.

The entire "way of remotion" can also be considered a limit, but a limit of the theological summation of all its constituent motions. Each motion, which is as a limit in the way considered above, is also an element in a super operation whose assembling is another limit, though one an order of magnitude higher.

34 Similarly, He must not be limited by any other terminators of form.

To consider one of these motions in the "way of remotion," we start with one earthly perfection. By removing all that limits it, we are left with the consideration of that perfection in the supereminent way in which it belongs to God. In order to do this, we must remove the infinitude of imperfections due to potency, matter, and the created nature of the thing. The process by which we do that is the same as that process by which we transcend the infinite in quantity, that is, it must be by a logical operation that can reach the terminus of this approach of knowledge.

The result of these logical operations, however, differ according to the different sciences. In theology, through the operation we reach God as existing in the supereminent way that we must attribute to the perfection we once held remote from Him. This perfection then distinguishes Him from, at least, one class of things that are not Him. In mathematics, however, the end result of the simplest analogous maneuver is infinite quantity, which does not strictly exist. It is only known as a privative, and hence cannot possess any of the perfections of the genus it is fundamentally related to. Thus God as an end of a theological process differs radically from the end of a mathematical limit.

As we said previously, the limit bears a kinship not just to one of these motions, but also to the sum of all such motion that is possible. Each one of these motions eliminates a class of creatures or beings from equivalency with God. Using the methodology of the limit once more, an infinity of these motions may be united into one super operation or motion. The limit of this composite motion is the complete way of remotion that Saint Thomas sets forth. In other words, when everything finite has been removed from God, we will be left with the Divine Substance Itself. But can we use a logical operation to traverse this theological summation, as we could if it were a mathematical operation?

In transferring the limit from the mathematical realm to the theological realm, we still hold the definition of the limit as **the logical terminus of an operation**. Our minds make a motion towards God, and this operation has a logical terminus – God Himself. But how can we know that God is this terminus? In mathematical operations, we know it, as we have said previously, from the underlying quality or nature. Analogously, the existence of God must underlie this motion towards Him.[35]

This apprehension of God as the limit is not held in the same way as the apprehension of the solution to a mathematical limit problem. The higher operation to which the solution of a mathematical operation appeals is a logical operation, but it is this very logical operation in a rational nature that cannot understand God directly. Thus, logic must have recourse to an operation that is higher still. To merely identify God as the logical terminus is possible through faith, but the operation which understands God directly can only be God's understanding of Himself.

Even the end of the "way of remotion" leaves us devoid of knowledge of "what He is in Himself." As Saint Thomas says, "when He is known as distinct from all things, we shall arrive at a proper consideration of Him. It will not, however, be perfect, because we shall not know what He is in Himself."[36] Thus, this approach, the "way of remotion," does indeed have a limit, and a limit that is reached. It turns out to be a "proper consideration" of God, but not a perfect one.

An Unreached Limit; The Names of God

Hence, God stands as a limit of our knowledge that is not reached. For though we can know each perfection as it is found in creatures at least in some way, how can we understand the mode in which those perfections exist in God? We certainly cannot know them perfectly in this second way. God, in Himself, is *not* a privative of His creation, as infinity is a privative of quantity. We cannot know God perfectly in this life; our intellects cannot comprehend God directly.[37] The motion of our knowledge towards Him is, like an

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35 Thus, "that God exists" is a proposition assumed by and prior to this thesis.
36 *Summa Contra Gentiles,* Chapter 14
37 Thus, we cannot in fact name Him perfectly from His effects. A sign of this is our diverse names for God; were we to know Him perfectly, we would only have one mode of knowing –

infinite mathematical operation, not sufficient in itself to reach the limit. It is this very insufficiency, along with the infinite remotion and subsequent infinite approach that allows the methodology of the limit to pertain to and aid Theology.

The approach of the knowledge of the human intellect to direct knowledge of God is exemplified by the names for God which *it provides*. The *mode of signifying* for these names of God is always defective; as Dionysius says, "such terms can be either affirmed or denied of God: affirmed, on account of the signification of the term; denied, on account of the mode of signification."[38] Thus, the human names of God are not capable of signifying His essence; that is, they cannot name Him according to His essence, nor convey the mode of His existence along with the name. They can only convey His existence according to the mode of His effects.[39] Hence, God irrevocably preserves His divine mysteriousness.

However, these human names do show how God stands as a limit to the "way of remotion" as a whole. In this way, He is reached as a limit in so far as the divine substance is pointed to logically, but not in so far as our understanding or application of the name extends to the essence of God. We name Him through names like the Highest Good, the First Being, and so on; we know the end of the operation, and name Him from it, but we do not understand. Neither the operation nor the normal leap of logical intuition that gives us the end of this theological investigation. As a logical operation is to the mathematical one it comprehends and surpasses, so must a higher operation be to the logical approach of God in the way of remotion. The substance of this higher operation must be God's knowledge of Himself, and this must remain outside man's understanding; yet the logical end of the operation is known to be God by faith. Thus, as a logical operation surpasses the infinity of the mathematical, so too in theology God's knowledge of Himself surpasses the infinity of the way of remotion.

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the mode in which He knows Himself – and correspondingly only one name for Him. As it is, we have many names for God, all pointing to Him logically; yet they are not synonyms.

38 *Summa Contra Gentiles,* Chapter 30, Paragraph 3

39 *Summa Contra Gentiles,* Chapter 30

Thus we see that this conclusion satisfies at least the basic inquiry, though it may fail to satisfy completely. If God be willing, we will exist at some point in the future with this higher operation completed in our intellects, not, indeed, by our own actions, but by God's benevolent grace and love, that we may share His knowledge of Himself in His one, pure, unlimited, infinite, and supreme Act.

Qual è 'l geomètra che tutto s'affige
per misurar lo cerchio, e non ritrova,
pensando, quel principio ond' elli indige,

tal era io a quella vista nova:
veder voleva come si convenne
l'imago al cerchio e come vi s'indova;

ma non eran da ciò le proprie penne:
se non che la mia mente fu percossa
da un fulgore in che sua voglia venne.

Appendix A: On the Aristotelean Limit

The project in this thesis is to not to discuss all the metaphysical meanings of the term "limit," but to discuss a meaning specific to the mathematician. Though it is a technical use of limit, this is not a project that would appeal to a specialist mathematician. Only a philosopher-mathematician, or mathematician-theologian, would try, as this thesis tries, to apply this meaning of limit to theology.

In Book V of the *Metaphysics,* Aristotle has a few lines that describe the senses in which he uses limit. He says,

> 17. "Limit" means
>
> (1) the ultimate part of each thing, or the first part outside of which no part can be found, or the first part inside of which all parts exist;
>
> (2) the form of a magnitude or of that which has magnitude;
>
> (3) the end of each thing, such being that towards which, but not that from which, a motion or action is directed, although sometimes it is both that from which and that towards which;
>
> (4) the final cause;
>
> (5) the substance of each thing, or the essence of each thing, for this is said to be the limit of knowledge; and if of knowledge then of the thing also.
>
> It is evident, then, that "limit" has as many senses as "principle," and yet more; for a principle is a limit, but not every limit is a principle.[40]

There are at least four distinct meanings here, but they do not all pertain to the investigation of the limit in Calculus. The first sense Aristotle mentions is easily related to this thesis: it is a general definition, but the one that belongs primarily to finite first dimension quantity, lines, in virtue of their endpoints. From there it extends to all "things," presumably in the widest sense possible, from cathedrals to politics to dance, and applies its meaning analogously, as each case dictates.

The third sense that Aristotle mentions seems directly related to our second sense of

40 *Aristotle's Metaphysics,* Book 5, Chapter 17. 1022a5-1022a13

limit, i.e. the limit of actual motion. It is the end of the motion, and Aristotle's sense allows for that limit to be “virtually contained” within the motion. As such, it completes our second sense of limit, but does not touch upon the third sense, the logical terminus of an operation. In a way, this is the consideration of the second sense of limit, but abstracted from the motion, yet this abstraction is not done in a way that leaves only the underlying limit – i.e. our first sense of limit – but rather a sense compounded logically of the first sense and a logical operation that utilized the nature of motion in the particular case.

The other two or three senses which Aristotle mentions, the form, the final cause, and the substance of each thing also do not complete the notion of the limit in our third sense. This is where the mathematician-theologian must pick up the technical sense of limit from the mathematician, and carry on alone into the realm of theology.

Appendix B: On the Modern Mathematical Limit

The limit in modern mathematics is nothing but a specialized term. As such, all generality has been defined away, and the philosopher-mathematician is at a loss. Origin, derivation, or extension are all denied, and the particular definition stands alone, isolated.

Here is a "formal definition" from Thomas/Finney's *Calculus*, 9th Edition.

> **Definition**
>
> **A Formal Definition of Limit**
>
> Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that $f(x)$ approaches the limit L as x approaches x_0, and write:
>
> $$\lim_{x \to x_0} f(x) = L$$
>
> if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x
>
> $$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

Thus, it is clear that this presentation of the limit is a befuddlement to the philosopher-mathematician or the mathematician-theologian, and less than useful for their ends. A speculative inquiry such as has been the subject of the thesis must not take this specialized formula as its jumping off point, but must appeal, as this thesis has attempted to do, to the physical and mathematical basis of what the limit actually is.

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The pseudo-quotation from Eüler on the cover page is taken from popular legend.

The closing quotation from Dante is taken from the original Italian text. It is the last three stanzas of the thirty-third and final canto of the *Paradiso*.